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GENERAL ATOMIC DIVISION
GENERAL DYNAMICS CORPORATION

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THE BEHAVIOR OF A THIN VISCOUS FILM UNDER MECHANICAL AND THERMAL FORCES

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Nuclear/Chemical Pulse Reaction Propulsion Project

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I. INTRODUCTORY REMARKS

The following treatment of the flow of a viscous, incompressible fluid over a flat plate under the influence of surface forces is designed to include at one time the presence of normal and tangential forces, and to deal with spatial and temporal behavior on an equivalent basis. It therefore serves to supplement and generalize the simpler, physically based account of Mautz* ~~(referred to hereafter as "M")~~. While it does not seem to lead to substantially new results, it enables consideration of more general spatial and temporal variations, without special artifices, and it illuminates the nature of the underlying approximations and their range of valid application.

The additional consideration of heat conduction and evaporation, carried out roughly in Section VI below, seems to offer a possible description of the experimental findings when combined with the viscous flow.

II. FORMULATION OF PROBLEM

The problem to be considered is that of a thin, uniform layer of a viscous, incompressible fluid ~~(of initial thickness δ)~~ covering an infinite, flat, horizontal plate, ~~(z = 0)~~, and is subjected to a cylindrically symmetric surface force. It is required to determine the motion of the surface:

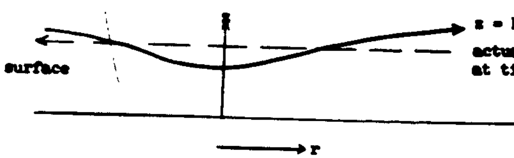

Fig. 1 Geometric Configuration

* GAMD-2237

The basic hydrodynamic equations are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} + \vec{g} \qquad (2.1)$$

$$\nabla\cdot\vec{v} = 0 \qquad (2.2)$$

where $\vec{v}$ is the velocity vector, p the pressure, ρ the density, μ the viscosity, $\nu = \mu/\rho$ the kinematic velocity, and $\vec{g}$ the external force field.

Since the fluid layer is thin, and the applied forces large and of short duration, it is justifiable to neglect the external (gravitational) force field $\vec{g}$. (There is no appreciable motion of the fluid due to $\vec{g}$ in the times of interest here.) In order to linearize the equations, the terms $(\vec{v}\cdot\nabla)\vec{v}$ will also be neglected; the validity of this will be considered later. The equations thus become

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{v} \qquad (2.3)$$

$$\nabla\cdot\vec{v} = 0 \qquad (2.4)$$

Introducing cylindrical coordinates, and taking note of the (assumed) symmetry, the basic equations are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \qquad (2.5)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w \qquad (2.6)$$

$$0 = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \qquad (2.7)$$

where $u = v_r$, $w = v_z$, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\,.$$

The boundary conditions are as follows:

At the plate, $z = 0$

$$u = w = 0\,. \qquad (2.8)$$

At the free surface, $z = h(r, t)$

$$f_z = -p + 2\mu\frac{\partial w}{\partial z} \qquad (2.9)$$

$$f_r = \mu\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right) \qquad (2.10)$$

where f_z, f_r are respectively the normal and tangential forces on the surface. The motion of the surface $z = h(r, t)$ is determined by

$$\frac{\partial h}{\partial t} = w \qquad (2.11)$$

III. SOLUTION IN TERMS OF POTENTIALS

These equations are most easily solved by the introduction of two potentials φ, ψ, defined by the relations

$$u = -\frac{\partial \varphi}{\partial r} - \frac{\partial \psi}{\partial z} \qquad (3.1)$$

$$w = -\frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \qquad (3.2)$$

Inserting these expressions into Eqs. (2.5), (2.6) and (2.7), one finds

$$\nabla^2\varphi = 0 \qquad (3.3)$$

from Eq. (2.7), and

$$\frac{\partial}{\partial r}\left(\frac{p}{\rho} - \frac{\partial \varphi}{\partial t}\right) = -\nu\frac{\partial}{\partial r}\nabla^2\varphi - \nu\frac{\partial}{\partial z}\left[\nabla^2\psi - \frac{\psi}{r^2} - \frac{1}{\nu}\frac{\partial \psi}{\partial t}\right]$$

$$\frac{\partial}{\partial z}\left(\frac{p}{\rho} - \frac{\partial \varphi}{\partial t}\right) = -\nu\frac{\partial}{\partial z}\nabla^2\varphi + \nu\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(\nabla^2\psi - \frac{\psi}{r^2} - \frac{1}{\nu}\frac{\partial \psi}{\partial t}\right)$$

from Eqs. (2.5) and (2.6), respectively. Since one may also place

$$\frac{\partial \varphi}{\partial t} = \frac{p}{\rho} \tag{3.4}$$

ψ must finally satisfy the equation

$$\nabla^2 \psi - \frac{\psi}{r^2} = \frac{1}{\nu}\frac{\partial \psi}{\partial t} \tag{3.5}$$

It is now convenient to introduce Laplace transforms in t, and Fourier-Bessel transforms in r. Writing

$$\Phi(z;k,s) = \int_0^\infty dr \int_0^\infty dt\; r\, J_o(kr) e^{-st}\, \varphi(z,r,t) \tag{3.6}$$

it follows that

$$\Phi'' - k^2 \Phi = 0 \tag{3.7}$$

$$s\Phi = \frac{P}{\rho} \tag{3.8}$$

where primes denote differentiation with respect to z, and

$$P(z;k,s) = \int_0^\infty dr \int_0^\infty dt\; rJ_o(kr) e^{-st} p(z,r,t) \tag{3.9}$$

Similarly, if

$$\Psi(z;k,s) = \int_0^\infty dr \int_0^\infty dt\; rJ_1(kr) e^{-st}\, \psi(z,r,t) \tag{3.10}$$

one finds

$$\Psi'' - m^2 \Psi = 0 \tag{3.11}$$

with

$$m^2 = k^2 + \frac{s}{\nu} \tag{3.12}$$

It should be noted that because of the structure of the original Eqs. (3.3) and (3.5), the F-B transform for φ is taken with respect to J_o, while that for ψ is taken with respect to J_1.

In the same way, one writes

$$U(z;k,s) = \int_0^\infty dr \int_0^\infty dt\; r\, J_1(kr) e^{-st} u(z,r,t) \tag{3.13}$$

$$W(z;k,s) = \int_0^\infty dr \int_0^\infty dt\; r\, J_o(kr) e^{-st} w(z,r,t) \tag{3.14}$$

and since $J_o'(x) = -J(x)$,

$$xJ_1'(x) = xJ_o(x) - J_1(x),$$

U and W are found to be given by

$$U = k\Phi - \Psi' \tag{3.15}$$

$$W = -\Phi' + k\Psi . \tag{3.16}$$

Solving Eqs. (3.7) and (3.11) subject to the boundary condition Eq. (2.8) yields

$$\Phi = A \cosh kz + B \sinh kz \tag{3.17}$$

$$\Psi = B \cosh mz + \frac{k}{m} A \sinh mz \tag{3.18}$$

where $A = A(k,s)$, $B = B(k,s)$ are to be determined from the boundary conditions Eqs. (2.9) and (2.10). Thus, if

$$F_z(z;k,s) = \int_0^\infty dr \int_0 dt\; r\, J_o(kr) e^{-st} f_z(z,r,t) \tag{3.19}$$

$$F_r(z;k,s) = \int_0^\infty dr\; r\, J_1(kr) e^{-st} f_r(z,r,t) \tag{3.20}$$

one finds

$$-(s+2\nu k^2)\Phi + 2\nu k\Psi' = \frac{1}{\rho}F_z \qquad (3.21)$$

$$2\nu k\Phi' - (s+2\nu k^2)\Psi = \frac{1}{\rho}F_r \qquad (3.22)$$

for $z = h$. Since h is ultimately a function of r and t, these equations are no longer linear (implicitly) and, in fact, their interpretation becomes obscure. However, if h is a smooth enough function of r and t, it is permissible to treat it as a constant and to regard the equations as linear. This point will be discussed further in the section of the validity of the various approximations. Inserting the expressions Eqs. (3.17) and (3.18) in Eqs. (3.21) and (3.21) and (3.22) and solving the resulting linear equations for A and B, one finds

$$\rho\Delta A = \left[-(s+2\nu k^2)\cosh mh + 2\nu k^2\cosh kh\right]F_z$$
$$-\left[-(s+2\nu k^2)\sinh kh + 2\nu km\sinh mh\right]F_r \qquad (3.23)$$

$$\rho\Delta B = -\left[-(s+2\nu k^2)\frac{k}{m}\sinh kh - 2\nu k^2\sinh kh\right]F_z$$
$$+\left[-(s+2\nu k^2)\cosh kh + 2\nu k^2\cosh mh\right]F_r \qquad (3.24)$$

where Δ is the determinant of the equations for A and B, and is given by

$$\Delta = -4\nu k^2(s+2\nu k^2) + (s+2\nu k^2)^2\left[\cosh kh\cosh mh - \frac{k}{m}\sinh kh\sinh mh\right]$$
$$+4\nu^2k^4\left[\cosh kh\cosh mh - \frac{m}{k}\sinh kh\sinh mh\right] \qquad (3.25)$$

U and W are given in terms of A and B by

$$W = (-k\sinh kz + \frac{k^2}{m}\sinh mz)A + (-k\cosh kz + k\cosh mz)B \qquad (3.26)$$

$$U = (k\cosh kz - k\cosh mz)A + (k\sinh kz - m\sinh mz)B \qquad (3.27)$$

It is not necessary to carry these expressions further in this degree of exactness, since in all cases of interest $kh \ll 1$ and $\nu k^2 \ll s$. In addition, $sh^2 \ll \nu$ for thin viscous films, and in most cases one even has $sh^2 < \nu$ for relatively thick viscous films. One may thus expand in series in terms of kh and $\nu k^2/s$, and retain only the leading terms. In some cases sh^2/ν may be treated in the same way, but for many purposes it is useful to retain the more exact functional dependence, in order to examine certain aspects of the temporal behavior of the solution. The reason for not introducing these simplifications earlier was to illustrate the exact form of some of the expressions, and to utilise the explicit z dependence of Eqs. (3.26) and (3.27) later for the examination of the validity of some of the approximations.

Thus one writes

$$\cosh kh \simeq 1$$
$$\cosh mh \simeq \cosh\sqrt{\frac{s}{\nu}}\,h = C(s)$$
$$\sinh kh \simeq kh$$
$$\sinh mh \simeq \sinh\sqrt{\frac{s}{\nu}}\,h = \sqrt{\frac{s}{\nu}}\,h\,S(s)$$

whence

$$\Delta(s) \simeq s^2C(s) \qquad (3.28)$$

$$A = \frac{1}{\rho sC(s)}\left[-C(s)F_z - kh(2S(s)-1)F_r\right] \qquad (3.29)$$

$$B = \frac{1}{\rho sC(s)}\left[kh\,S(s)F_z - F_r\right] \qquad (3.30)$$

and eventually

$$W(h) = \frac{1}{\rho sC(s)}\left[\left[C(s)-S(s)\right]k^2h\,F_z - \left[C(s)-1\right]kF_r\right] \qquad (3.31)$$

$$U(h) = \frac{1}{\rho sC(s)}\left[(1-C(s))\,kF_z + \frac{sh}{\nu}S(s)\,F_r\right] \qquad (3.32)$$

If only leading terms in sh^2/ν are retained, these equations reduce to

$$W(h) = \frac{h^3}{3\mu}k^2F_z - \frac{h^2}{2\mu}kF_r \qquad (3.33)$$

$$U(h) = -\frac{h^2}{2\mu}kF_z + \frac{h}{\mu}F_r \qquad (3.34)$$

one finds

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and eventually

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with

$$P(h) = - F_z - kh\, F_r \qquad (3.25)$$

It is now necessary to transform back to the variables r and t. In the first place, if

$$G(k) = \int_0^\infty dr\, r\, J_0(kr)\, g(r)$$

then

$$kG(k) = \int_0^\infty dr\, r\, J_1(kr)(- g'(r))dr$$

$$k^2G(k) = \int_0^\infty dr\, r\, J_0(kr)(- g''(r) - \frac{g'(r)}{r})\, dr ,$$

while if

$$\bar{G}(k) = \int_0^\infty dr\, r\, J_1(kr) g(r)$$

then

$$k\bar{G}(k) = \int_0^\infty dr\, r\, J_0(kr)(g'(r) + \frac{g(r)}{r}) .$$

Thus,

$$k^2F_z \text{ transforms to } - (\frac{\partial^2 f_z}{\partial r^2} + \frac{1}{r}\frac{\partial f_z}{\partial r})$$

$$kF_r \text{ transforms to } \frac{\partial f_r}{\partial r} + \frac{1}{r} f_r$$

$$kF_z \text{ transforms to } - \frac{\partial f_z}{\partial r}$$

$$F_r \text{ transforms to } f_r$$

The temporal transforms may be determined by reference to tables of Laplace transforms, and are as follows:

$$\frac{1}{sC(s)} \text{ transforms to } 1 - \frac{4}{\pi} \sum_0^\infty \frac{(-1)^k}{2k+1} e^{-\pi^2(k + \frac{1}{2})^2 t\, \nu/h^2} = 1 - \alpha_1(t)$$

and

$$\frac{S(s)}{sC(s)} \text{ transforms to } 1 - \frac{8}{\pi^2} \sum_0^\infty \frac{1}{(2k+1)^2} e^{-\pi^2(k + \frac{1}{2})^2 t\nu/h^2} = 1 - \alpha_2(t)$$

The functions $\alpha_1(t)$ and $\alpha_2(t)$ have the following properties:

$$\alpha_1(o) = \alpha_2(o) = 1$$

$$\alpha_1(\infty) = \alpha_2(\infty) = 0$$

For $t > h^2/\nu$

$$\alpha_1(t) \simeq \frac{4}{\pi} e^{-\frac{\pi^2 t \nu}{4h^2}}$$

$$\alpha_2(t) \simeq \frac{8}{\pi^2} e^{-\frac{\pi^2 t \nu}{4h^2}}$$

For small t

$$\alpha_1(t) \simeq 1 - 2 \sqrt{\frac{t\nu}{h^2}}\, e^{-\frac{h^2}{4\nu t}}$$

$$\alpha_2(t) \simeq 1 - c \left(\frac{t\nu}{h^2}\right)^{\frac{3}{2}} e^{-\frac{h^2}{4\nu t}} \qquad \text{(c constant)}$$

If f_z and f_r are the product of spatial and temporal functions, q_z, q_r, then for instance

$$w(h,t) = \frac{h}{\mu}(-f_z'' - \frac{1}{r} f_z') \int_0^t \alpha_2(\tau) q_z(t - \tau)d\tau - \frac{1}{\mu}(f_r' + \frac{1}{r}f_r)\int_0^t \alpha_1(\tau) q_r(t-\tau)d\tau \qquad (3.36)$$

If the time of application of the surface forces is long compared to h^2/ν, then Eq. (3.33) applies, and this reduces to

$$w(h,t) = - \frac{h^3}{3\mu}(f_z'' + \frac{1}{r} f_z')q_z(t) - \frac{h^2}{2\mu}(f_r' + \frac{1}{r}f_r)q_r(t) . \qquad (3.37)$$

In most cases $q_2(t)$ and $q_r(t)$ may be taken to be the same function.

Taking note of Eq. (2.11), one therefore has

$$\frac{dh}{dt} = -(h^3 a_z + h^2 a_r)\frac{q(t)}{\mu} \tag{3.38}$$

where

$$a_z = \frac{1}{3}(f''_z + \frac{1}{r}f'_z)$$

and

$$a_r = \frac{1}{2}(f'_r + \frac{1}{r}f_r)$$

are functions of r only. The solution of Eq. (3.38), subject to the condition $h = \delta$ for $t = 0$, is found to be given by

$$\int_0^t \frac{q(t)dt}{\mu} = \frac{1}{a_r\delta}(\frac{\delta}{h} - 1) - \frac{a_z}{a_r^2}\log\left[\frac{a_r}{a_r + a_z\delta}(\frac{\delta}{h} - 1) + 1\right] \tag{3.39}$$

In particular, for $a_r = 0$, this reduces to

$$\frac{1}{h^2} - \frac{1}{\delta^2} = 2\int_0^t \frac{q(t)dt}{\mu}$$

i.e.,

$$\frac{\delta^2}{h^2} = 1 + 2\delta^2 a_z \int_0^t \frac{q(t)dt}{\mu} \tag{3.40}$$

On the other hand, if $a_z = 0$, the solution becomes

$$\frac{\delta}{h} = 1 + \delta a_r \int_0^t \frac{q(t)}{\mu}dt \tag{3.41}$$

From Eq. (3.38) or (3.39) it can be shown simply that

$$\delta(a_r + \delta a_z)\int_0^t \frac{q(t)dt}{\mu} \geq \frac{\delta}{h} - 1 \geq \delta a_r \int_0^t \frac{q(t)dt}{\mu} \tag{3.42}$$

The implications of these formulae will be discussed later.

IV. VALIDITY OF APPROXIMATIONS

It is clear from Eqs. (3.31) through (3.34) that $\left|W(h)/U(h)\right| = O(kh)$ so that U (and hence u) is the predominant velocity component. (Here k is used interchangeably as a transformation variable and as a scale factor for radial variations, i.e., $u'(r)/u(r) \sim k$.) When f_z is predominant, one has

$$\left|\frac{u^2}{p}\right| \sim k^2h^4 \left|\frac{f_z}{\mu^2}\right|$$

which is small except for unreasonably large values of $\left|f_z\right|$. If f_r is predominant, this becomes

$$h^2 \left|\frac{f_r}{\mu^2 k}\right|$$

which is small for modest values of f_r, but becomes large for large f_r.

Similarly, one has

$$\frac{k|u|^2}{\mu} \left|\nabla^2 u - \frac{u}{r^2}\right| \sim k^2h^4 \left|\frac{f_z}{\mu s}\right|$$

which is again small except for extremely large values of $\left|f_z\right|$, (or

$$kh^2 \left|\frac{f_r}{\mu s}\right|$$

if f_r is predominant), so that the non-linear terms are again negligible. Thus the neglect of the non-linear terms is certainly justified for the conditions which obtain for this problem.

Finally, the expressions given for h (Eqs. 3.39 through 3.41) indicate that the approximations made following Eq. (3.22) are valid provided either

(1) t is small
(2) δ is small
(3) f_z and f_r do not have abrupt discontinuities or violent oscillations.

Conditions (2) and (3) together are generally satisfied sufficiently to validate the procedure used, which is then analogous to "adiabatic" perturbation of the differential equations in both space and time.

V. DISCUSSION: VISCOUS EQUATIONS

It is not proposed to give a detailed numerical discussion of the basic formulae Eqs. (3.39) through (3.41), since this has been done comprehensively in the paper by Mauts (M) cited earlier, which hews closely to the experimental situation. It does, however, seem desirable to correlate the final results given here with those in M, and to add a few additional comments about their general behavior.

The general equation (3.38) governing the surface motion (h) contains contributions from both normal and tangential forces. It may be noted that when one or the other of these forces is neglected, the resulting equation corresponds exactly to the continuity equation (M 3) (or M 16), with however the neglect of a term involving the spatial variation of h. This neglect is a consequence of the "adiabatic" approximation made following Eqs. (3.21) and (3.22) and discussed in the previous section, and seems justified for the situation of interest here. The resulting formulae

$$\frac{\delta^2}{h^2} = 1 + 2\delta^2 a_z \int_0^t \frac{g(t)}{\mu}\, dt \tag{3.40}$$

$$\frac{\delta}{h} = 1 + \delta a_r \int_0^t \frac{g(t)}{\mu}\, dt \tag{3.41}$$

which apply in the presence of normal or tangential forces respectively are the exact counterparts of (M 14) and M 17), where

$$a_z = \frac{1}{3}\left(f''_z + \frac{1}{r} f'_z\right)$$

f_z being the normal force ($' = d/dr$), and

$$a_r = \frac{1}{2}\left(f'_r + \frac{1}{r} f_r\right).$$

[$g(t)$ has been taken identically 1 in M.]

It is of some interest to consider, at least briefly, the relation between these two cases. It is clear, first, that the film moves more rapidly under the influence of a tangential force than under a normal one. If one equates the surface height for the two conditions (normal and tangential force alone), and assumes a scale length L along the surface, so that $a_r \sim f_r/L$, $a_z \sim f_z/L^2$, one finds that for small times one must have

$$f_r \simeq \frac{\delta}{L} f_z \tag{5.1}$$

for equal displacements, while for large times

$$f_r \simeq \sqrt{\frac{2\mu}{t}}\; f_z \tag{5.2}$$

Formula (5.1) is understandable physically by noting that the tangential force need only move a thickness δ initially, while the normal force has to move an amount determined by its scale length L . The interpretation of (5.2) is not as clear. In both cases the greater sensitivity of the motion to tangential forces is clear.

The final correlation that need be made between the two approaches concerns the temporal decay of any disturbance. In this treatment the behavior follows directly from the functions $\alpha_1(t)$ and $\alpha_2(t)$ described following (3.35). It is immediately evident that the temporal part of the motion has the leading term

$$e^{-\frac{\mu\pi^2}{4a_s^2}t} \tag{5.3}$$

in agreement with (M 21).

It is of significance to note that the fractional motion of the surface is very sensitive to the absolute value of the initial height. Thus, the "halving" time is

$$t \simeq \frac{3\mu}{a_s\delta^2} \tag{5.4}$$

and

$$t \sim \frac{\mu}{a_r \delta} \qquad (5.5)$$

in the case of a normal and a tangential force respectively, δ being the initial thickness. Taking the normal force case, with $t = 10^{-4}$ sec, $\mu = 10$ poise, peak pressure $= 2 \times 10^9$ dynes/cm^2 and scale length 5 cm, one finds

$$\frac{\delta}{h} = 1.002 \qquad \text{for} \qquad \delta = 2 \times 10^{-3} \text{ cm}$$

but

$$\frac{\delta}{h} = 2.45 \qquad \text{for} \qquad \delta = 7 \times 10^{-2} \text{ cm.}$$

This mechanism is therefore adequate to describe the substantial motion of a thick film in a short time, but indicates only a very small motion of a thin film in a similar time. Experimentally it has been found, however, that some portions of the film are sometimes completely removed, the position of these portions being reasonably correlated with the regions of major surface force variations as described by the functions a_z (and a_r). This leads one to surmise that a combination of phenomena is taking place, of which the viscous flow is only one. The following section describes another effect, which together with the viscous flow seems adequate to describe the observed results, at least qualitatively.

VI. THERMAL CONDUCTION AND EVAPORATION

In the following it is supposed that the temperatures associated with the high surface forces are such as to cause at least partial evaporation of the viscous film, and a rough calculation is attempted to estimate the rate at which this may occur.

$$z = 0 \qquad T = T_o$$

$$z = \zeta \qquad T = T_v$$

$$z = \infty \qquad T = T_\ell$$

The initial surface of the film is assumed at $z = 0$, while the liquid-vapor interface is at $z = \zeta$ (a function of time). It is assumed that the time scale is such that the vapor remains on top of the liquid as shown during the period in question, and that the liquid is infinite in extent. (These two points will be discussed below.)

λ = heat of vaporization
ρ = density
c = heat capacity
k = thermal conductivity
T_v = vaporization temperature
T_ℓ = liquid temperature (at ∞—at plate)
a^2 = $k/\rho c$

The subscript "1" refers to the vapor, "2" to the liquid.

The equations governing the temperature variation are

$$\frac{\partial T_1}{\partial t} = a_1^2 \frac{\partial^2 T_1}{\partial z^2} \qquad z < \zeta \text{ (vapor)}$$

$$\frac{\partial T_2}{\partial t} = a_2^2 \frac{\partial^2 T_2}{\partial z^2} \qquad z > \zeta \text{ (liquid)} \qquad (6.1)$$

The boundary conditions are

$$T_1 = T_o \qquad z = 0 \qquad (6.2)$$

$$T_1 = T_2 = T_v \qquad z = \zeta \qquad (6.3)$$

$$-\lambda \rho \frac{d\zeta}{dt} = \left(k_1 \frac{\partial T_1}{\partial z} - k_2 \frac{\partial T_2}{\partial z}\right) \qquad z = \zeta \qquad (6.3)$$

$$T_2 = T_\ell \qquad z = \infty \qquad (6.5)$$

The solutions have the form (cf "Die Differential und Integralgleichungen der Mechanik und Physik" by P. Frank and R. V. Mises, VII, p.565)

$$T_1 = A_1 + B_1 \operatorname{erf}\left(\frac{z}{2a_1\sqrt{t}}\right)$$

$$T_2 = A_2 + B_2 \operatorname{erf}\left(\frac{z}{2a_2\sqrt{t}}\right) . \qquad (6.6)$$

Then

$$A_1 = T_o \qquad (6.7)$$

$$A_2 + B_2 = T_\ell \qquad (6.8)$$

$$A_1 + B_1\psi_1 = A_2 + B_2\psi_2 = T_v \qquad (6.9)$$

where

$$\psi_i = \operatorname{erf}\left(\frac{z}{2a_i\sqrt{t}}\right) \qquad i = 1, 2$$

Since the boundary conditions are independent of t, one must have

$$\zeta = \alpha\sqrt{t} \qquad (6.10)$$

and it remains to determine α.

It is clear that

$$B_1 = -\frac{(T_o - T_v)}{\psi_1}$$

$$B_2 = -\frac{(T_v - T_\ell)}{(1 - \psi_2)}$$

$$A_2 = T_\ell - B_2 = T_1 - B_2\psi_2 .$$

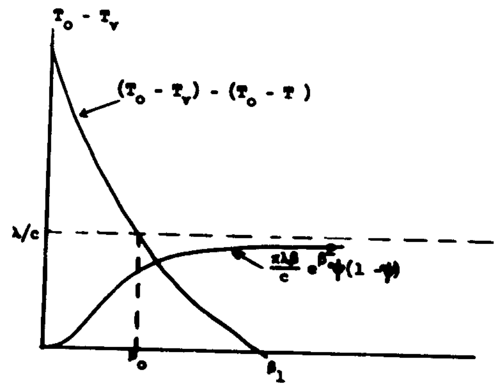
The boundary condition (6.6) then leads to

$$-\frac{\sqrt{\pi}}{2}\lambda\alpha = -c_1a_1\frac{T_o - T_v}{\psi_1}e^{-\frac{\alpha^2}{4a_1^2}} - c_2a_2\frac{T_v - T_\ell}{1 - \psi_2}e^{-\frac{\alpha^2}{4a_2^2}} \qquad (6.11)$$

For the conditions under which the vapor is highly compressed, it seems adequate, as an approximation, to take $a_1 = a_2$ $c_1 = c_2$ $\rho_1 = \rho_2$ (at least initially.) Writing $\beta = \alpha/2a$, the determining equation for β becomes

$$-\frac{\sqrt{\pi}\,\lambda\beta}{c} = -\frac{T_o - T_v}{\psi}e^{-\beta^2} + \frac{T_v - T_\ell}{1 - \psi}e^{-\beta^2}$$

Multiplying through by $\psi(1 - \psi)e^{\beta^2}$, one notes that the solution must lie between β_o and β_1 as shown in the figure



Since $T_v - T_\ell << T_o - T_2$, while $\lambda/c(T_v - T_\ell)$ is of the order of 1, one may easily convince oneself that it is permissible to use the asymptotic form of ψ, i.e.,

$$1 - \psi \simeq e^{-\beta^2} / \sqrt{\pi}\,\beta .$$

Thus,

$$(T_o - T_\ell)e^{-\beta_1^2} = \sqrt{\pi}\,\beta_1(T_v - T_\ell) \qquad (6.12)$$

$$(T_o - T_\ell)e^{-\beta_o^2} \simeq \sqrt{\pi}\,\beta_o\left[(T_v - T_\ell) + \lambda/c\right] \qquad (6.13)$$

Taking

$$T_v - T_\ell = .01\,(T_o - T_\ell) \tag{6.14}$$

one finds

$$1.65 < \beta < 1.65 \tag{6.15}$$

so that

$$\alpha = 3.5\ a$$

Putting

$$k = 10^{-3}$$
$$\rho = 1$$
$$c = .4$$

leads to

$$a = .05$$

and hence to

$$\zeta = .18\sqrt{t}\ . \tag{6.16}$$

For $t = 10^{-4}$ this yields

$$\zeta = 1.8\times10^{-3}\ \underline{cm} \tag{6.17}$$

so that nearly one mil of the liquid could be evaporated in this time!

It remains to discuss the approximations introduced initially. Taking the liquid to be infinitely deep is too optimistic an assumption. The presence of a supporting plate, possibly of high heat capacity, will tend to reduce the motion of ζ (the liquid-vapor interface.) On the other hand, assuming the vapor to remain completely in place is much too pessimistic. The surface forces will tend to move the upper layer of the vapor and thus expose the liquid-vapor interface to a higher heat flux, tending to compensate for the presence of the supporting plate.

A final remark concerns the viscosity of the film which decreases with increasing temperature, resulting in more rapid motion under the surface forces.

It thus seems that for a thin film the combination of thermal flux and surface forces could result in the viscous liquid disappearing entirely in regions of high pressure variation.

TT:rl;db

9/

END